S6 PHYSICS NOTES

SIMPLE HARMONIC MOTION (SHM)

This is a periodic motion whose acceleration is directly proportional to the displacement from the fixed point and is directed towards that point

CHARACTERISTICS OF SIMPLE HARMONIC MOTION

- It is a periodic motion
- Acceleration is directly proportional to the displacement from a fixed point
- Acceleration is directed towards a fixed point
- Mechanical energy of a body undergoing free SHM is constant

TERMS AND DEFINITIONS

1. **Equilibrium position**: Is a point where the vibrating particle would come to rest if it were to lose all its energy
2. **Displacement:** Is the distance moved by the vibrating particle away from an equilibrium position in a given direction
3. **Amplitude:** Is the maximum displacement of the vibrating particles from the rest (mean) position
4. **Frequency (*f*):** Is the number of vibrations made per second.
5. **Period *(T)*:** Is the time taken to complete one vibration
6. **Phase angle *(Φ)*:** Is the angle by which one vibrating particle leads or lags the other particle
7. **Angular velocity (*ω*):** Is the angle swept by a vibrating particle in one second

NB: (i) For particles vibrating in phase with SHM, the equation for displacement is given by: $x = a\sin\omega t$; *a* is amplitude and *ω* is the angular velocity

(ii) If the particles vibrate with a phase angle Φ, then the equation becomes $x = a\sin(\omega t \pm \phi)$

EXPRESSION FOR VELOCITY OF SHM

For particles vibrating in phase with SHM, $x = a\sin\omega t$

$$\Rightarrow \sin\omega t = \frac{x}{a}; v = a\omega\cos\omega t$$

$$\text{but}; \sin^2\omega t + \cos^2\omega t = 1$$

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{v}{a\omega}\right)^2 = 1$$

$$\Rightarrow v = \pm\omega\sqrt{\left(a^2 - x^2\right)}$$

When the particle passes the equilibrium position, x = 0, hence; $v = \pm\omega a$ (This is the maximum velocity attained)

At the extreme ends of the motion, x = *a* hence the velocity of the particle is zero

EXPRESSION FOR ACCELERATION OF SHM

For particles vibrating in phase with shm, $x = a\sin\omega t$

$$v = a\omega\cos\omega t$$

$$a = -a\omega^2\sin\omega t(\text{but...}x = a\sin\omega t)$$

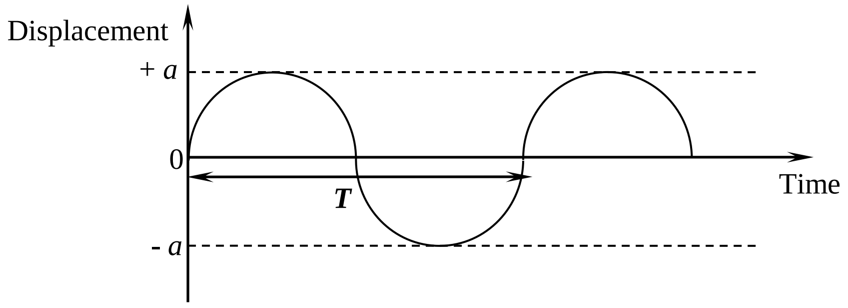
$$a = -\omega^2 x(\text{negative..means..acceleration..is..directed..towards..a..fixed..point})$$

Maximum acceleration = $\omega^2 a$

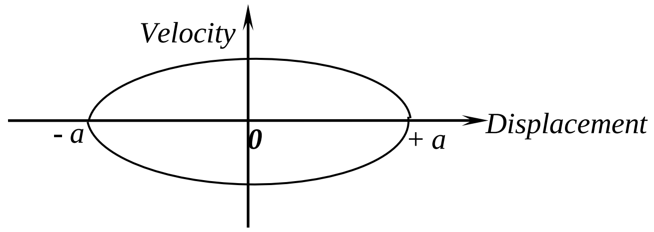
Force on a body; F $= ma = m\omega^2 a$ **; Maximum force F_{max}** $= m\omega^2 a$

GRAPHS OF SIMPLE HARMONIC MOTION

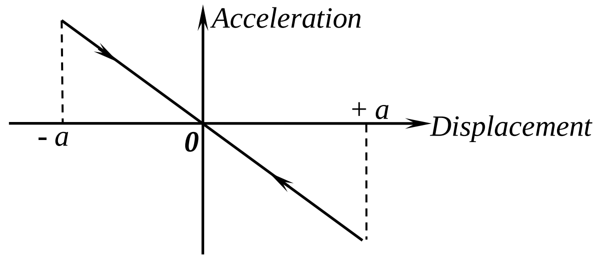
(a) Displacement – time graph



(b) Velocity – displacement graph



(c) Acceleration – displacement graph



Numerical examples

1. A particle is moving with S.H.M of period 8 seconds and amplitude 5.0m. Calculate
   (a) The angular velocity
   (b) The frequency of the motion
   (c) The maximum acceleration of the particle
   (d) Speed when the particle passes the equilibrium position
2. A particle performing S.H.M has a speed of $3ms^{-1}$ and $1.4ms^{-1}$ when at 2m and 2.4m respectively from the fixed point O. calculate:
   (a) Amplitude and angular velocity
   (b) Maximum speed and maximum acceleration
   (c) The speed of the particle when it is 1m from O
3. Displacement, $X = 8\cos\left(\frac{\pi}{2}t + \frac{\pi}{3}\right)$ of a particle moving in a straight line
   (a) Show that the particle performs simple harmonic motion
   (b) State the amplitude, phase angle, angular velocity and period of the motion

(c) Find the maximum speed of the motion and the maximum acceleration of the particle

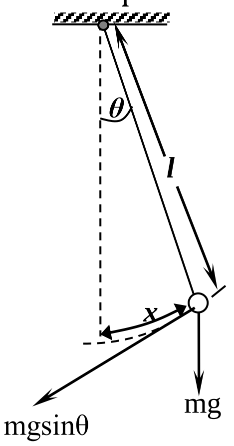
Examples of motions that approximate SHM

1. A simple pendulum
2. Mass on a helical spring
3. Floating cylinder
4. A gas in a cylinder with a movable frictionless piston
5. Liquid oscillating in U – tube

A simple pendulum

Consider a particle of mass, **m** tied to a string of length, *l* and given a very small displacement, θ and released to oscillate

It performs simple harmonic motion as shown below



Restoring force; $F = -mg\sin\theta$ *but F = ma hence; ma = -mgsinθ*

$a = -g\sin\theta$; $\sin\theta = \frac{x}{l}$ *but for very small angle in radians;* $\sin\theta \approx \theta$

$a = -g\frac{x}{l} \Rightarrow a = -\left(\frac{g}{l}\right)x$

Of the form $a = -\omega^2 x$ *hence it performs simple harmonic motion*

Period and frequency of oscillation of the pendulum

$$\omega^2 = \frac{g}{l} \Rightarrow \omega = \sqrt{\frac{g}{l}}; (\omega = 2\pi f) \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{l}{g}}$$

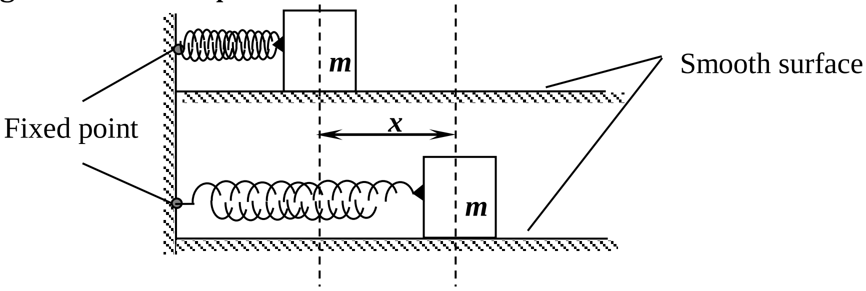
Approximations made;

- The amplitude of oscillation is very small
- The simple pendulum oscillates in a vacuum
- Friction at the support is negligible

Mass on a helical spring:

(a) **The horizontal vibrating spring**

Consider a spring of force constant, ***k*** fixed at one end and attached to a body of mass, ***m*** placed on a smooth table. When the particle is given a small displacement, ***x*** and released, the resulting motion is simple harmonic



After a displacement, the restoring force $F = -kx$ But $F = ma$ $ma = -kx \Rightarrow a = -\left(\frac{k}{m}\right)x$; of the form $a = -\omega^2 x$ hence it performs simple harmonic motion.

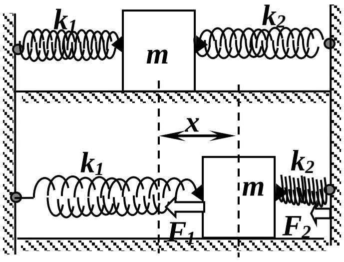
$$\omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}; and....T = 2\pi\sqrt{\frac{m}{k}}$$

Approximations made

- The effective mass of the spring is negligible
- The surface is very smooth and so there is negligible resistance to oscillation
- The amplitude of oscillation is very small

(b) **Two horizontal parallel springs**

If springs S_1 and S_2 having force constants $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ are arranged in parallel to a mass, $\boldsymbol{m}$ and placed on a smooth horizontal surface and the mass is given a small displacement on one side and released, it performs SHM



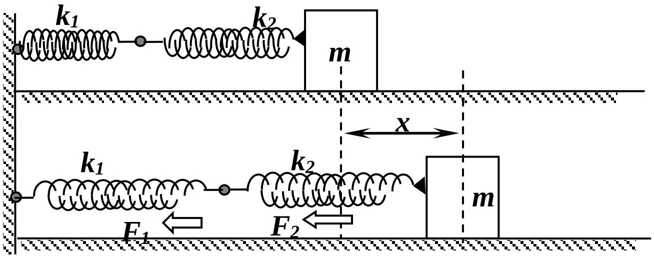
Restoring force; $F = -(F_1 + F_2) = -(k_1 x + k_2 x) = -(k_1 + k_2)x$

but $F = ma \Rightarrow ma = -(k_1 + k_2)x \Rightarrow a = -\left(\frac{k_1 + k_2}{m}\right)x$. This is of the form $a = -\omega^2 x$, hence it performs SHM

$$\omega^2 = \frac{k_1 + k_2}{m} \Rightarrow \omega = \sqrt{\frac{k_1 + k_2}{m}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k_1 + k_2}{m}} \text{ and}$$

$$T = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

(c) **Two springs arranged in series**



Each spring extends by a different amount such that $x = x_1 + x_2$ ....(i)

Note that the same restoring force acts on the springs; $F_1 = F_2 = F$

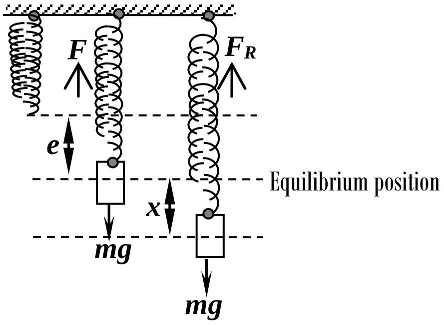
$$x_1 = \frac{F}{k_1}, x_2 = \frac{F}{k_2} and; x = \frac{F}{k} \Rightarrow \frac{F}{k} = \frac{F}{k_1} + \frac{F}{k_2} \Rightarrow k = \frac{k_1 k_2}{k_1 + k_2} (k..is..effective..force..constant)$$

$$F = -kx \Rightarrow ma = -\left(\frac{k_1 k_2}{k_1 + k_2}\right)x \Rightarrow a = -\frac{k_1 k_2}{m(k_1 + k_2)}x$$

Which is of the from a = - $\omega^2 x$ hence it performs simple harmonic motion

Assignment: Derive equations for period T and frequency, f of the vibration of the mass above

(d) The vertical vibrating spring



A spring of force constant, ***k*** is fixed at one end and a mass, ***m*** is suspended from the other end to cause an extension, ***e***. If a mass is slightly displaced downwards and released to oscillate, it performs simple harmonic motion

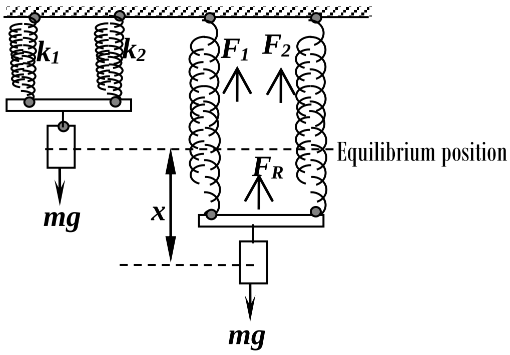
At equilibrium; $ke = mg \Rightarrow \frac{m}{k} = \frac{e}{g}$ ....(a) Note this relation

After a displacement, the restoring force $F_R = -(k(e + x) - mg)$

$\Rightarrow ma = -kx \Rightarrow a = -\left(\frac{k}{m}\right)x$; of the form $a = -\omega^2 x$ hence it performs simple harmonic motion.

Assignment; Derive equations for period, T and frequency, f of the oscillation of the spring above

(e) Two parallel vertical springs



At equilibrium position; $(k_1 + k_2)e = mg$ ....(a)

After a displacement, the restoring force,

$$F_R = -((F_1 + F_2) - mg)$$

$$\Rightarrow ma = -((k_1 + k_2)(e + x) - mg)$$

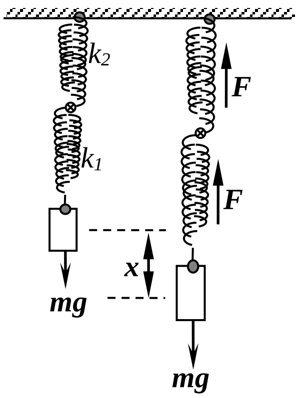
$$a = -\left(\frac{k_1 + k_2}{m}\right)x$$

Which is an equation of the form $a = -\omega^2 x$ hence it performs simple harmonic motion

Assignment: Derive expressions for period, T and frequency, f of the oscillation of the above springs.

Two vertical springs in series:

If the springs arranged in series are suspended vertically and a mass is attached to the free end of one of them, they perform simple harmonic motion when slightly displaced.



When displaced slightly and released, the springs attain difference extensions such that

$$x = x_1 + x_2 \Rightarrow \frac{F}{k} = \frac{F}{k_1} + \frac{F}{k_2} \Rightarrow k = \frac{k_1 k_2}{k_1 + k_2}$$

$$F = -kx \Rightarrow ma = -\left(\frac{k_1 k_2}{k_1 + k_2}\right)x \Rightarrow a = -\frac{k_1 k_2}{m(k_1 + k_2)}x$$

$$a = -\omega^2 x$$

Assignment: Derive expressions for period, T and frequency, f of the oscillation of the springs above

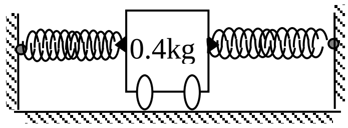
Numerical examples

1. A horizontal spring of force constant 200Nm^{-1} fixed at one end has a mass of 2kg attached to the free end and rests on a smooth horizontal table. The mass is pulled through a distance of 4.0cm and released. Calculate the;

(i) Angular speed of oscillation (10rads^{-1})
(ii) Maximum velocity attained by the vibrating body (0.4ms^{-1})
(iii) Acceleration when the body is half way towards the centre from its initial position (2ms^{-2})

2. A spring of force constant 5Nm^{-1} is placed horizontally on a smooth table. One end of the spring is fixed and a mass ***X*** of 0.20kg is attached to the free end. ***X*** is displaced through a distance of 4mm along the table and then released
   (i) Show that the motion of ***X*** is simple harmonic
   (ii) Determine the period of oscillation (1.257s)
   (iii) Find the acceleration at the extreme end of the spring (0.1ms^{-2})
   (iv) Find the maximum kinetic energy of the spring (4.0x10^{-5}J)

3. A trolley of mass 0.4kg is attached to two fixed positions as shown below

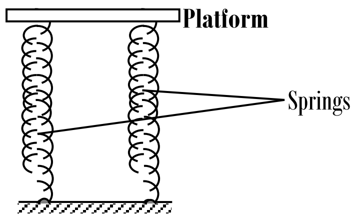


When it is displaced horizontally 0.20m from equilibrium position and released, it performs simple harmonic motion of period 4.0s. Ignoring any friction effect, calculate the;
   (i) Maximum speed of the trolley (0.314ms^{-1})
   (ii) Maximum kinetic energy of the trolley (1.97x10^{-2}J)
   (iii) Effective force constant of the springs (0.987Nm^{-1})

4. A mass of 0.1kg is attached to a free end of a vertical helical spring whose upper end is fixed to a spring extended by 0.04m. It is now pulled downwards a distance of 0.02m and then released. Determine;
   (i) Its period (0.40s)
   (ii) The maximum force acting on it during the oscillation (0.4905N)
   (iii) The kinetic energy when the body passes through the mean position (1.56x10^{-3}J)

5. A metal cylinder of mass 0.2kg is attached to the lower end of a light helical spring which is fixed to the upper platform and a spring extends by 0.16m. The metal cylinder is now pulled down a further 0.08m.
   (i) Find the force that must be exerted to keep it there (2.943N)
   (ii) The cylinder is then released, find the period of oscillation and the kinetic energy when it passes through equilibrium position (0.802s, 0.03928J)

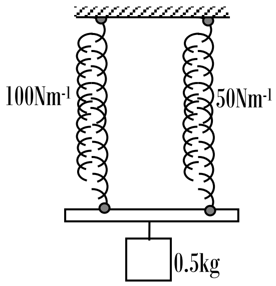
6. A light platform is supported by identical springs, each having a spring constant 2Nm^{-1} as shown below



   (a) Calculate the weight which must be attached on the platform in order to produce a depression of 3.0cm (0.12N)
   (b) If the weight remains on the platform and the platform is displaced a further 1.0cm and released
   (i) What is the frequency of oscillation of the platform (4.07Hz)

(ii) What is the maximum acceleration of the platform (6.5396ms^{-2})

7. A mass of 0.5kg is suspended from the free end of two springs of force constants 100Nm^{-1} and 50Nm^{-1} as shown below



Calculate (i) the extension produced (0.0327m)

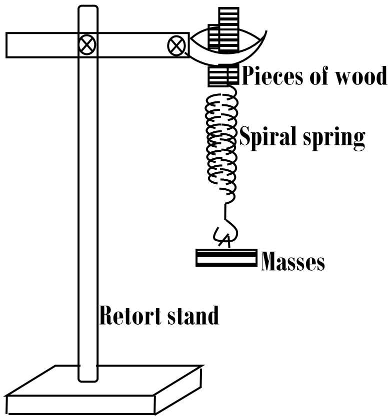
(ii) The tension in each spring (3.27N, 1.635N)

(iii) The energy stored in the combined springs (0.080J)

(iv) The frequency of the small oscillation of the mass if it is given a small vertical displacement and released

(27.2Hz)

Experiment to determine the force constant, *k* of a spiral spring



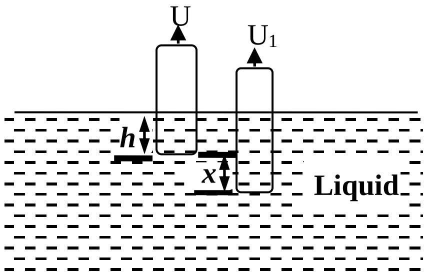
A known mass, ***m*** is suspended from a spiral spring and is given a very small downward displacement and released to oscillate.
The stop clock is started and the time for 20 oscillations is read and recorded hence the period, ***T*** is calculated.
The procedure is repeated for different masses, ***m*** and the values tabulated including, T^2
A graph of T^2 against ***m*** is plotted and its slope ***s*** determined
The force constant of the spiral spring is calculated from $k = \frac{4\pi^2}{s}$

Errors of the experiment

- The effect of air resistance and other dissipative forces reduce the accuracy of the experiment
- Error due to the mass of the spring not being negligible as assumed in the theory of the experiment
- Errors due to large amplitude of vibration

Floating cylinder

Consider a cylinder of cross sectional area, ***A*** and mass, ***m*** floating in a liquid of density, ***ρ*** with a height, ***h*** submerged as shown below



At equilibrium; $U = mg \Rightarrow m = \rho Ah$ ……… (i)
After displacement, the restoring force;

$$F = -(\rho A(h+x)g - mg)$$

$$\Rightarrow ma = -\rho Agx$$

$$\Rightarrow a = -\frac{\rho gA}{\rho Ah}x \Rightarrow a = -\frac{g}{h}x$$

Which is an equation of the form $a = -\omega^2 x$ hence it performs simple harmonic motion

NB: *Viscous drag offered by the liquid on to the cylinder causes damping, the cylinder moves with reducing amplitude and the oscillation dies out in a short time.*

Assignment: Derive equations for period, ***T*** *and frequency,* ***f*** *of the oscillation*

Numerical examples

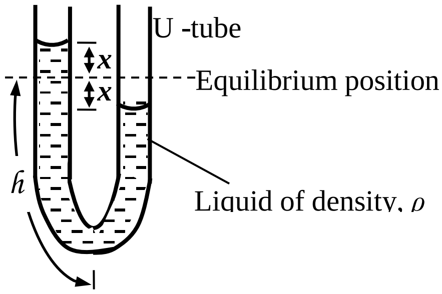
1. A test tube of mass 10.0g is floated in water by placing a mass of 10g at its bottom. The tube is depressed slightly into water and released;
   (i) Show that the test tube performs simple harmonic motion
   (ii) If the external diameter of the test tube is 20cm, find the period of the resulting motion (0.091s)
2. (a)A uniform cylindrical rod length 0.08m, cross sectional area, 0.02m² and density 900kgm⁻³ floats in a liquid of density 1000kgm⁻³. The rod is depressed through a distance of 0.005m and released
   (i) Show that the rod performs simple harmonic motion
   (ii) Find the frequency of the resulting oscillation
   (1.585Hz)

(iii) Find the velocity of the rod when it's a distance 0.004m above the equilibrium position (0.035ms^{-1})

(b) Explain briefly why the oscillations die out.

Oscillation of a liquid in a U – tube

Consider a column of a liquid of density, ρ contained in an open U – tube of uniform cross sectional area, **A**. Let **h** be the height of each limb at equilibrium. Suppose the liquid on one side is depressed through, **x** by blowing down air, a liquid performs simple harmonic motion.



At equilibrium, mass of the liquid; $M = 2Ah\rho$ ………… (i)

After a displacement, excess pressure; $p = 2x\rho g$

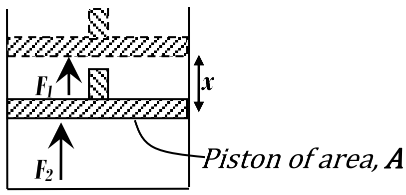
Restoring force; $F_R = -pA = -2x\rho gA$

$\Rightarrow ma = -2x\rho gA \Rightarrow a = -\dfrac{2x\rho gA}{2Ah\rho} \Rightarrow a = -\left(\dfrac{g}{h}\right)x$ which is of the form, $a = -\omega^2 x$ hence the liquid performs simple harmonic motion.

Assignment: Derive an equation for the period, T and frequency, f of oscillation of the liquid.

A piston in a gas filled vessel

Consider a gas of volume ***V*** at a pressure, ***p*** contained in a cylinder of cross sectional area, ***A*** by a frictionless piston of mass, ***m***. When the piston is pushed down through a small distance, ***x*** and released, it performs simple harmonic motion.



At equilibrium; $F_1 = pA$ …….(i)

If the piston is displaced downwards so that the pressure of the gas increases by Δp as volume reduces by $\Delta V = Ax$ then

Force acting on the piston; $F_2 = (p + \Delta p)A$ ……(ii)

Restoring force; $F = -(F_2 - F_1) = -(pA + \Delta pA - pA) = -\Delta pA$ ….(iii)

If the displacement is very small, the compression occurs at constant temperature and therefore obeys Boyle's law

$pV = (p + \Delta p)(V - \Delta V) \Rightarrow pV = pV - \Delta Vp + \Delta pV - \Delta p\Delta V; but \Delta p.\Delta.V \approx 0$

$\Rightarrow \Delta pV = \Delta Vp \Rightarrow \Delta p = \dfrac{\Delta Vp}{V}$ …..(iv)

Using equations (iii) and (iv), then $F = -\dfrac{\Delta VAp}{V} = -\dfrac{A^2 px}{V} \Rightarrow a = -\left(\dfrac{A^2 p}{mV}\right)x$ which is of the form $a = -\omega^2 x$ hence the oscillation is simple harmonic.

Assignment: Show that the period of oscillation of the piston is given by $T = \dfrac{2\pi}{A}\sqrt{\dfrac{mV}{p}}$ *and hence derive the equation for the frequency of oscillation.*

Energy of the oscillator

Assume a body of mass, ***m*** moving with S.H.M of amplitude, ***a***, angular velocity, $\boldsymbol{\omega}$ with a linear speed, **v**

If x is the displacement of the body at any time, t, given by $x = a\sin\omega t$

(a) **Kinetic energy**

$$K.E = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\omega^2\left(a^2 - x^2\right) = \tfrac{1}{2}m\omega^2\left(a^2 - a^2\sin^2\omega t\right) = \tfrac{1}{2}m\omega^2 a^2\cos^2\omega t$$

(b) **Potential energy**

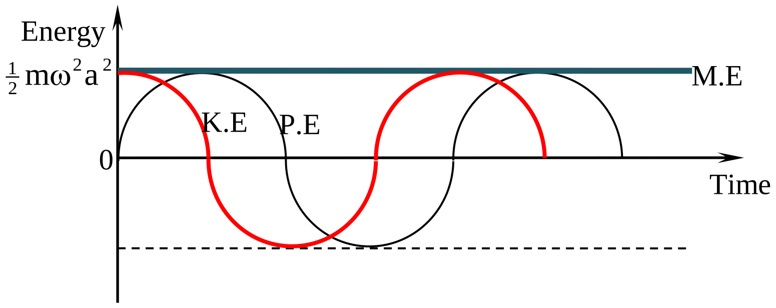
$$PE = \int_0^x m\omega^2 x\,dx = \tfrac{1}{2}m\omega^2 x^2 = \tfrac{1}{2}m\omega^2 a^2\sin^2\omega t$$

(c) **Mechanical energy (Total energy)**

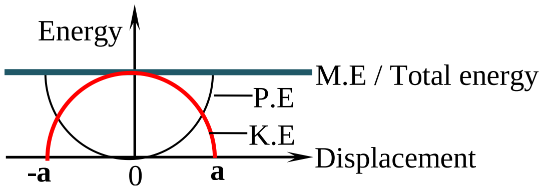
$$M.E = P.E + KE = \tfrac{1}{2}m\omega^2 a^2\left(\sin^2\omega t + \cos^2\omega t\right) = \tfrac{1}{2}m\omega^2 a^2$$

For a freely oscillating system, mechanical energy remains constant.

A graph of Energy against time for an oscillating system



A graph of Energy against displacement of oscillating system



Practical examples of simple harmonic motion

1. Motion of pistons in engines.
2. Motion of a balance wheel in a pendulum clock.

TYPES OF OSCILLATIONS

There are basically three types of oscillations namely

1. Free oscillation
2. Damped oscillation
3. Forced oscillation

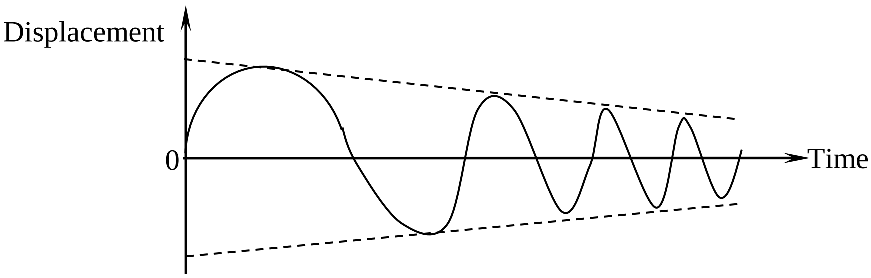
1. **Free oscillation**

This is an oscillation which occurs in the absence of any dissipative forces such as air resistance, viscous drag and friction. Here the amplitude and the total mechanical energy remain constant and the system continues indefinitely with its natural frequency of oscillation.

2. **Damped oscillation;**

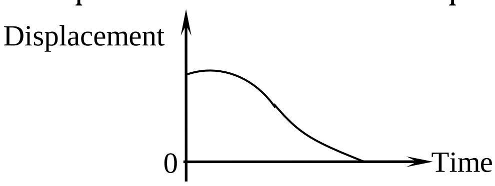
This is an oscillation where energy is taken away from the system to do external work against dissipative forces. The amplitude reduces with time and the oscillation ultimately dies out.

(a) **Under damped oscillation:** This is where the system actually oscillates but the oscillations gradually die out due to dissipative forces.



Examples of such oscillations are; A simple pendulum set to oscillate in air, oscillation of a mass in horizontal and vertical springs etc

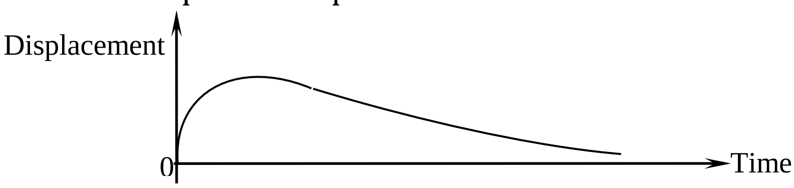
(b) **Critically damped oscillations:** This is where the system does not oscillate when displaced but returns to its equilibrium position in a minimum possible time



Examples include: shock absorbers in cars, toilet doors/ bank doors.

(c) **Over damped oscillation**

This is where the system does not oscillate when displaced but takes a long time to return to the equilibrium position



Examples include: mass on horizontal spring moving over a rough surface, body oscillating in a viscous fluid.

3. **Forced oscillation:**

This is an oscillation driven by an external force. The system oscillates at the same frequency as that of the driving force.

Examples include: oscillation of a building in an earth quake, oscillation of a string of a guitar when plucked, oscillation of air column in piped instruments.